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The exact solution of two new types of Schrödinger equation

Harold Exton
'Nyuggel', Lunabister, Dunrossness, Shetland ZE2 9JH, UK

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Abstract. A compact solution of a special biconfluent Heun equation is obtained by means of a Laplace transform. The eigenvalues and eigenfunctions of two new types of Schrödinger equation are deduced by appropriate variable transformations. The associated potential function in each case can be arranged to have a double well.

This study is motivated by obtaining a hitherto un-noticed compact solution of a non-trivial special case of the second-order linear differential equation

$$xy'' + (a_1x^2 + b_1x + c_1)y' + (a_0x^2 + b_0x + c_0)y = 0 \tag{1}$$

from which the solutions of two new types of Schrödinger equation can be deduced.

In (1), put $y = Y'$, and obtain

$$xY''' + (a_1x^2 + b_1x + c_1)Y'' + (a_0x^2 + b_0x + c_0)Y' = 0. \tag{2}$$

Take the Laplace transform

$$Y = \int_C \exp(xt)\, u(t)\, \mathrm{d}t \tag{3}$$

where the modulating function $u(t)$ is determined by the differential equation

$$\begin{aligned} t(a_1t + a_0)u'' - [t^3 + b_1t^2 + (b_0 - 4a_1)t - 2a_0]u' \\ +[(c_1 - 3)t^2 + (c_0 - 2b_1)t - 2a_1 - b_0]u = 0. \end{aligned} \tag{4}$$

If any non-trivial solution of (4) can be obtained, then the solution of (2), and, in turn, (1), follows from (3), with a suitable choice of the contour of integration C. In general, (4) is probably as difficult to solve as (1), but if we put

$$c_1 = 3 \qquad c_0 = 2b_1 \qquad \text{and} \qquad b_0 = 2a_1 \tag{5}$$

(4) reduces to the first-order equation

$$t(a_1t + a_0)v' = [t^3 + b_1t^2 - 2a_1t - 2a_0]v \tag{6}$$

where $v = u'$. Hence,

$$v = \exp[\tfrac{1}{2}t^2/a_1 + (b_1/a_1 - a_0/a_1^2)t]t^{-2}(t + a_0/a_1)^{a_0^2/a_1^3 - b_1a_0/a_1^2}. \tag{7}$$

The contour of integration in (3) may be taken to be a simple loop beginning and ending at $-\infty$ if $\mathrm{Re}(x) > 0$, or at ∞ if $\mathrm{Re}(x) < 0$, and encircling the origin once. This approach has been suggested by Murphy (1960) pp 141 and 214, from a strictly practical point of view. The reader should also consult Ince (1926), ch 8 and 18.

The modulating function $u(t)$ is obtained from (7) by indefinite integration with respect to t, and in the present context, the most convenient form of carrying out this process is by replacing t by $s - a_0/a_1$, when, apart from arbitrary constants,

$$u(s) = \int_0^s \exp(\tfrac{1}{2}s^2/a_1 + sb_1/a_1)(s - a_0/a_1)^{-2} s^{a_0^2/a_1^3 - b_1 a_0/a_1^2}\, \mathrm{d}s. \qquad (8)$$

If the integrand is expanded as a triple series, term-by-term integration within its domain of convergence gives

$$u(s) = \sum \frac{(2a_1)^{-m}(b_1/a_1)^n (a_0/a_1)^p (2, p) s^{1+a_0^2/a_1^3 - b_1 a_0/a_1^2 + 2m+n+p}}{m!n!p!(1 + a_0^2/a_1^3 - b_1 a_0/a_1^2 + 2m + n + p)} \qquad (9)$$

where the indices of summation in this study run over all the non-negative integers. As usual, the Pochhammer symbol (a, m) denotes the product $a(a+1)(a+2)\ldots(a+m-1) = \Gamma(a+m)/\Gamma(a)$; $(a, 0) = 1$. From (3),

$$Y(x) = \exp(-a_0 x/a_1) \int_C \exp(xs)\, u(s)\, \mathrm{d}s \qquad (10)$$

where we recall that $t = s - a_0/a_1$, so that the right-hand member of (10) becomes, formally,

$$\exp(-a_0 x/a_1) \sum \frac{(2a_1)^{-m}(b_1/a_1)^n (a_0/a_1)^p (2, p)}{m!n!p!(1 + a_0^2/a_1^3 - b_1 a_0/a_1^2 + 2m + n + p)}$$
$$\times \int_C \exp(xs)\, s^{1+a_0^2/a_1^3 - b_1 a_0/a_1^2 + 2m+m+p}\, \mathrm{d}s = \exp(-a_0 x/a_1) F(x), \text{ say.} \qquad (11)$$

When evaluated, the inner integral of (11) is found to be proportional to

$$x^{-2-a_0^2/a_1^3 + b_1 a_0/a_1^2 - 2m-n-p} / \Gamma(-1 - a_0^2/a_1^3 + b_1 a_0/a_1^2 - 2m - n - p) \qquad (12)$$

where terms with negative exponents vanish, so that if

$$1 + a_0^2/a_1^3 - b_1 a_0/a_1^2 = -N \qquad N = 0, 1, 2, \ldots \qquad (13)$$

a polynomial form of $F(x)$ follows. Apart from any constant factors, the corresponding solution of (2) with the conditions (5) is

$$Y(x) = \exp(-a_0 x/a_1) \sum \frac{(2a_1)^{-m}(-b_1/a_1)^n (-a_0/a_1)^p (2, p)}{m!n!p!}$$
$$\times (-N, 2m + n + p) x^{N-2m-n-p}. \qquad (14)$$

As a simple consequence of the binomial theorem, the previous result may be expressed as the following double series:

$$\exp(-a_0 x/a_1) \sum \frac{(2a_1)^{-m}(-a_0/a_1)^p (2, p)(-N, 2m + p)(x + b_1/a_1)^{N-2m-p}}{m!p!} \qquad (15)$$

which may be regarded as a combination of the Hermite polynomial and a special case of the Charlier polynomial. In this case, when (13) holds, the previous formal process is justified.

We now consider (1) with conditions (5):

$$xy'' + (a_1 x^2 + b_1 x + 3)y' + (a_0 x^2 + 2a_1 x + 2b_1)y = 0 \qquad (16)$$

from which we obtain its normal form

$$W'' + [-\tfrac{1}{4}a_1^2 x^2 + (a_0 - \tfrac{1}{2}a_1 b_1)x - \tfrac{1}{4}b_1^2 + \tfrac{1}{2}b_1 x^{-1} - \tfrac{3}{4}x^{-2}]W = 0 \qquad (17)$$

by replacing the dependent variable of (16) by

$$\exp(-a_1x^2 - \tfrac{1}{2}b_1x)x^{-1/2}W. \tag{18}$$

If $x = z^q$ and after replacing W by

$$z^{\frac{1}{2}(q-1)}w \tag{19}$$

it follows that

$$w'' + \{q^2[-\tfrac{1}{4}a_1^2z^{4q-2} + (a_0 - \tfrac{1}{2}a_1b_1)z^{3q-2} - \tfrac{1}{4}b_1^2z^{2q-2} + \tfrac{1}{2}b_1z^{q-2}] + (\tfrac{1}{4} - q^2)z^{-2}\}w = 0. \tag{20}$$

The parameters a_1 and a_0 can be so arranged that a_1^2 and $a_0 - \frac{1}{2}a_1b_1$ are completely free. Hence, *independent eigenvalues are associated with the coefficients of* z^{4q-2} *and* z^{3q-2}, and on putting $q = \frac{1}{2}$ or $q = \frac{2}{3}$, we have, respectively

$$w'' + \tfrac{1}{4}[-\tfrac{1}{4}a_1^2 + (a_0 - \tfrac{1}{2}a_1b_1)z^{-\frac{1}{2}} - \tfrac{3}{4}b_1^2z^{-1} + \tfrac{1}{2}b_1z^{-3/2}]w = 0 \tag{21}$$

with eigenvalue a_1, *and*

$$w'' + \{4/9[-\tfrac{1}{4}a_1^2z^{2/3} + a_0 - \tfrac{1}{2}a_1b_1 - \tfrac{1}{4}b_1^2z^{-2/3} + \tfrac{1}{2}b_1z^{-4/3}] - 7z^{-2}/36\}w = 0 \tag{22}$$

with eigenvalue a_0.

The eigenvalue equation in each case is given by (13), namely

$$1 + a_0^2/a_1^3 - b_1a_0/a_1^2 = -N \qquad N = 0, 1, 2, \ldots. \tag{23}$$

Equation (14) furnishes the eigenfunctions, noting that $y = Y'$, making the appropriate changes of variable outlined above. Both (21) and (22) could conceivably be useful, in that each of the associated potential functions could be made to include a double well on the half line, with a singularity of fractional order. It might also be noted that the form of (1) considered in this study is a new type of biconfluent Heun equation for which a compact solution has been deduced. See Exton (1991) and Ronveaux (1995) for example.

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